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Quantum Hall Effect and Noncommutative Geometry

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1 Introduction

Our aim is to introduce the ideas of noncommutative geometry through the example of the Quantum Hall Effect (Q.H.E.).We present a few concrete situations where the concepts of noncommutative geometry find physical applications.

The Quantum Hall Effect [1][2][3][4] is a remarkable example of a purely experimental discovery which "could" have been been predicted because the tools required are not extremely sophisticated and were known at the time of the discovery. What was missing was a good understanding topological rigidity produced by the quantum mechanics whose consequences can be tested at a macroscopic level: The quantized integers of the conductivity are completely analogous to the topological numbers one encounters in the study of fiber-bundles.

One can give a schematic description of the quantum Hall effect as follows. It deals with electrons constrained to move in a two dimensional semiconductor sample in a presence of a applied magnetic field perpendicular to the sample. Due to the magnetic field, the Hilbert space of an electron is stratified into Landau Levels separated by an energy gap (called the cyclotron frequency and proportional to the applied field). Each Landau level has a macroscopic degeneracy given by the area of the sample divided by a quantum of area (inversely proportional to the field) equal to $2\pi l^2$ where the length l is the so called magnetic length. It is useful to think of the magnetic length as a Plank constant $l^2 \sim \hbar$. The limit of strong magnetic field is very analogous to a classical limit.

The electrons behave much like incompressible objects occupying a quantum of area. Thus, when their number times $2\pi l^2$ is exactly equal a multiple of the area, it costs the energy gap to add one more electron. This discontinuity in the energy needed to add one more electron is at the origin of the incompressibility of the electron fluid. The number of electrons occupying each unit cell is called the filling factor, and the transverse conductivity is quantized each time the filling factor is exactly an integer.

We shall stick to this simple explanation, although this cannot be the end of the story. Indeed, if it was correct, the filling factor being linear in the magnetic field, the quantization of the conductance should be observed only at specific values of the magnetic field. In fact, it is observed on regions of finite width called plateaux, and it is necessary to invoke the impurities and localized states to account for these plateaus. Roughly speaking, some of the states are localized and do not participate to the conductance. These states are populated when the magnetic field is in a plateau. We refer the reader to a previous Poincare seminar for an introduction to these effects [4]. What is important for us here is that (although counter intuitive) it is possible to realize experimentally situations where the filling factor is *exactly* an integer (or a fraction as we see next).

It came as a great surprise (rewarded by the Nobel prize[1]) when the Quantum Hall Effect was observed at non-integer filling factors which turn out to always be simple fractions. To explain these fractions, it was necessary to introduce some very specific wave functions and to take into account the interactions between the electrons. The proposed wave functions are in some sense variational, although they carry no adjustable free parameters.

From a historical perspective, this approach of a universal phenomena through the introduction of "rigid" trial wave functions was going in opposite direction of the renormalization group ideas blossoming at the time. Nevertheless, the trial wave functions are undoubtedly the most powerful tools available at present and it remains a challenge to reconcile them with the field theoretical point of view. The recent progress in understanding the renormalizability of noncommutative field theories [6] may be a crucial step in this direction.

Another contact point between field theory and the Q.H.E. must be mentioned. Remarkably, many good trial wave functions for the fractional Q.H.E. are also correlation functions of conformal field theory (CFT) and integrable models [7]. This relation is not fully understood. The possibility to identify the certain fields such as the currents with the electron arises from the fact that their operator product expansion are analytical, a property obeyed by the wave functions when electrons approach each other. In some sense, the short distance properties combined with the constraint arising from incompressibility, control the topological structure of the theory. Another mysterious aspect is that the quasiparticles of the Hall effect carry a fractional charge, as if an electron breaks up into pieces. A very similar phenomena occurs in CFT where the current can be obtained as the short distance expension of other fields. In the study of the XXZ spin chain, the magnon is known to break up into two spinons. This leads to rich variety of phenomena which can be studied by exact methods. J.M. Maillet explains these aspects in this seminar.

The noncommutative-geometrical aspects that occupy us here are preeminent in the fractional Q.H.E. when the lowest Landau Level is partially filled. A great simplification and a source of richness comes from the Lowest Landau Level (LLL) projection. When the energy scales involved are small compared to the cyclotron gap, one can study the dynamics by restricting it to the LLL. As a consequence of this drastic reduction of the degrees of freedom, the two coordinates of the plane obey the same commutation relations as the position and the momentum in quantum mechanics. The electron is therefore not a point like particle anymore and can at best be localized at the scale of the magnetic length.

In a series of beautiful experiments [11][20], it was realized that the electron behaves as a neutral particle when the filling factor is exactly $1/2$ [12]. We shall present an image of the $\nu = 1/2$ state where electrons are dressed by a companion charge of the opposite sign. When the filling factor is $1/2$, the two charges conspire to make a neutral bound state with the structure of a dipole [16]. It is this very specific experimental situation which we advocate to be a paradigm of noncommutative geometry [22]. We hope to convince the reader that it is deeply connected to the non commutative field theory aspects developed by V. Rivasseau in this seminar.

We also review the example of the Skyrmion [30] which are the noncommutative analogue of the non linear sigma model solitons. This gives an exactly solvable model where the classical concept of winding number has a quantum counterpart which is

[1]Horst L. Störmer, Daniel C. Tsui. Robert B. Laughlin, in 1998.

simply the electric charge of the soliton. This gives a physical application for the noncommutative geometry developed by A.Connes [36]. In particular, the topological invariant which measures the winding number has a noncommutative analogue which evaluates the electric charge of the Skyrmion [32].

A remarkable aspect of the Q.H.E. physics is that it becomes a matrix theory. The study of this theory has suscitated many very interesting works, in particular in relation with the Chern-Simon theory [8]. These aspects have been extensively studied [9] by A. Polychronakos who develops them in this seminar.

2 Lowest Landau Level physics

2.1 Single particle in a magnetic field

Let us first recall some basic facts about the motion of a particle in a magnetic field. We consider a charge q particle in the plane subject to a magnetic field B transverse to the plane.

It is convenient to define the magnetic length l by:

$$l = \sqrt{\frac{1}{qB}}. \tag{1}$$

To simplify the notations, we take units where the magnetic length is equal to one: $l = 1$.

We introduce a vector potential A for the magnetic field:

$$1 = \partial_x A_y - \partial_y A_x \tag{2}$$

The vector potential A is defined up to a gauge transformation $A \to A + \nabla\chi$. The action from which the equations of motion of a mass m and charge 1 particle (confined to the plane) in presence of the magnetic field $B\hat{\boldsymbol{z}}$ derive, is given by:

$$S = \int (\frac{m}{2}\dot{r}^2 - A\dot{r})\ dt \tag{3}$$

Using the canonical rules, we obtain a Hamiltonian:

$$H_0 = \frac{1}{2m}\left(\mathbf{p} + \mathbf{A}\right)^2 = \frac{\boldsymbol{\pi}^2}{2m}, \tag{4}$$

where $p_i = m\partial_{x_i}/i - A_i$ is the momentum conjugated to x_i, the so-called dynamical momenta:

$$\pi_x = p_x + A_x, \quad \pi_y = p_y + A_y \tag{5}$$

obey the commutation relations:

$$[\pi_i, \pi_j] = i\epsilon_{ij}, \quad [r_i, r_j] = 0, \quad [\pi_i, r_j] = -i\delta_{ij}, \tag{6}$$

where ϵ_{ij} is the antisymmetric tensor $\epsilon_{xy} = -\epsilon_{yx} = 1$.

If we define creation and annihilation operators as linear combinations of the two dynamical momenta :

$$a = \sqrt{\frac{1}{2}}(\pi_x + i\pi_y), \quad a^+ = \sqrt{\frac{1}{2}}(\pi_x - i\pi_y), \tag{7}$$

obeying the Heisenberg relations:

$$[a, a^{+}] = 1. \tag{8}$$

The Hamiltonian is:

$$H_0 = \frac{1}{2m}(a^{+}a + \frac{1}{2}). \tag{9}$$

Its spectrum is that of an oscillator:

$$E_n = \frac{1}{2m}(n + \frac{1}{2}), \tag{10}$$

with $n \geq 0$. Each energy branch is called a Landau level.

A useful gauge is the so-called symmetric gauge defined by:

$$A_x = -\frac{y}{2}, \quad A_y = \frac{x}{2}. \tag{11}$$

In this gauge

$$a = \frac{i}{\sqrt{2}}(\partial_{\bar{z}} + z), \quad a^{+} = \frac{i}{\sqrt{2}}(\partial_z - \bar{z}), \tag{12}$$

We can define new coordinates R_x, R_y which commute with the dynamical momenta:

$$R_x = \frac{x}{2} - p_y, \quad R_y = \frac{y}{2} + p_x, \tag{13}$$

with the commutator given by:

$$[R_i, R_j] = i\epsilon_{ij}, \quad [\pi_i, R_j] = 0. \tag{14}$$

The coordinates so defined are called guiding centers. the guiding center coordinates can be combined into two oscillators:

$$\begin{aligned} b^{+} &= \frac{1}{\sqrt{2}}(R_x + iR_y) = \frac{1}{\sqrt{2}}(z - \partial_{\bar{z}}), \\ b &= \frac{1}{\sqrt{2}}(R_x - iR_y) = \frac{1}{\sqrt{2}}(\bar{z} + \partial_z). \end{aligned} \tag{15}$$

The lowest Landau level wave functions are obtained upon acting onto the ground state of (9) with $(b^{+})^m$. In this gauge, the angular momentum L is a good quantum number and they carry an angular momentum $L = -m$. Their expression is proportional to:

$$\Psi_m(z) = z^m \exp\left(-z\bar{z}\right), \tag{16}$$

and they can be visualized as thin circular shells of radius $\sqrt{\frac{m}{2}}$ around the origin. Thus if we quantize the system in a a disk of finite radius R, we recover the expected degeneracy (17) by keeping only the wave functions confined into the disk $m \leq m_0 = 2R^2$.

The fact that the guiding center coordinates commute with H_0 implies that its spectrum is extremely degenerate. The two coordinates R_x, R_y do not commute with each other and cannot be fixed simultaneously. There is a quantum uncertainty

$\Delta R_x \Delta R_y = 1$ to determine the position of the guiding center. Due to the uncertainty principle, the physical plane can be thought of as divided into disjoined cells of area 2π where the guiding center can be localized. The degeneracy per energy level and per unit area is $1/2\pi$ so that in an area Ω, the number of degenerate states is:

$$N_\Omega = \frac{\Omega}{2\pi}, \tag{17}$$

so that electrons behave "as if " they acquire some size under a magnetic field, the area being inversely proportional to B.

In the strong magnetic field limit, one projects the dynamics onto the lowest Landau level LLL $n = 0$. In other words, we impose the constraint $a|\text{sates}\rangle = 0$ and the dynamics is fully controlled by the guiding center coordinates.

2.2 Noncommutative product

Let us show how the noncommutative product on functions arises from the projection in the LLL. For future convenience, we keep the charge of the particle equal to q in this section.

The idea is to transform a function $f(x)$ into a one body operator: $\int d^2x|x\rangle$ $f(x)\langle x|$, and to project this operator in the LLL:

$$\hat{f} = \sum_{n,m} |n\rangle \int \langle n|x\rangle f(x) \langle x|m\rangle d^2x \ \ \langle m|, \tag{18}$$

where n, m are the indices of the LLL orbitals. In this way, we transform a function into a matrix.

Conversely, using coherent states, a one body operator acting in the LLL can be transformed into a function. The coherent states $|z\rangle$ are the most localized states in the LLL. They are the adjoint of the state $\langle z|$ defined by $\langle z|n\rangle = \psi_n(z)$ where $\psi_n(z)$ are the LLL wave functions (16). They form an overcomplete basis, and transform a matrix into a function by:

$$f(z, \bar{z}) = \langle z|\hat{f}|z\rangle. \tag{19}$$

To see this work in practice, it is convenient to use the symmetric gauge. In the symmetric gauge the LLL degenerate wave functions carrying an angular momentum l are given by:

$$\langle z|l\rangle = (q^{1/2}z)^l/(2\pi l!)^{1/2} e^{-qz\bar{z}/2}. \tag{20}$$

The wave functions are normalized so that: $\langle l|l'\rangle = \delta_{ll'}$. The parameter q has the dimension of a length^{-2}. In the limit $q \to \infty$ where we recover the classical limit.

The guiding center coordinates (34) are combined into oscillators acting within the LLL:

$$b = q^{-1}\partial_z + \bar{z}/2, \ b^+ = -q^{-1}\partial_{\bar{z}} + z/2. \tag{21}$$

It is convenient to absorb the factor $e^{-qz\bar{z}/2}$ In the measure. Thus, we recover the Bargman Fock representation of the operators on analytical functions:

$$b = q^{-1}\partial_z, \ b^+ = z. \tag{22}$$

They obey the commutation relation: $[b, b^+] = q^{-1}$ and in the limit where $q \to \infty$ they become true coordinates.

The coherent states $|\xi\rangle$ are defined as the eigenstates of b: $b|z\rangle = \bar{\xi}|\xi\rangle$. Thus, their wave function is given by:

$$\langle z|\xi\rangle = e^{q\bar{\xi}z}, \tag{23}$$

and their scalar product $\rho = \langle z|z\rangle = q/\pi$ does not depend on $|z\rangle$. The Q-symbol [5] of a one body operator $\hat{A} = \sum_{ll'} |l\rangle \hat{A}_{ll'} \langle l'|$ acting within the LLL consists in bracketing it between coherent states and normalizing it by ρ:

$$a(z, \bar{z}) \quad = \quad \langle z|\hat{A}|z\rangle/\rho. \tag{24}$$

In particular one has:

$$e^{iPX} = e^{i(\bar{p}z+p\bar{z})} = \langle z|e^{i\bar{p}b^+}e^{ipb}|z\rangle/\rho. \tag{25}$$

Therefore, the Q-symbol induces a non commutative product on functions which we denote by $*$:

$$a * b = \langle z|\hat{A}\hat{B}|z\rangle/\rho. \tag{26}$$

If we apply this to the plane waves, we obtain the product:

$$e^{iPX}e^{iRX} = e^{-\frac{P.R}{q}}e^{i\frac{P\wedge R}{q}}e^{i(P+R)X}. \tag{27}$$

This algebra is known as the magnetic translation algebra [17] and plays an important role in the theory of the Q.H.E.. We shall see next that it originates from the fact that the coordinate X has a dipolar structure.

In the limit $q \to \infty$, the *-product coincides with the ordinary product. Using (25) one can evaluate the first order correction to the ordinary product given by:

$$a * b = ab + \frac{1}{\pi\rho}\partial_{\bar{z}}a\partial_z b + O(\frac{1}{\rho^2}) \tag{28}$$

It is straightforward to establish the following dictionary between the commutative space and non commutative LLL projected space:

$$\frac{1}{\pi}\int\ .\ d^2x \to \frac{1}{q}\mathrm{Tr}\ ., \quad \partial_{\mathrm{z}}. \to \mathrm{q[b,.]}, \quad \partial_{\bar{\mathrm{z}}}. \to -\mathrm{q[b^+,.]} \tag{29}$$

where Tr now stands for the trace of the matrix in the LLL Hilbert space.

3 Interactions

3.1 Spring in a magnetic field

Let us first consider a simple model for particles in interaction. A pair of particles of opposite charge $\pm q$ are coupled by a spring. In the Landau gauge:

$$A_x = 0, \quad A_y = x, \tag{30}$$

their dynamics follows the Lagrangian:

$$L = (x_1\dot{y}_1 - x_2\dot{y}_2) - k/2((x_1 - x_2)^2 + (y_1 - y_2)^2) \tag{31}$$

where in (31) we have have taken the strong B field limit which enables to neglect the masses of the particles. The Hamiltonian is therefore:

$$H = \frac{k}{2}((x_1 - x_2)^2 + (y_1 - y_2)^2) = P^2/2k \tag{32}$$

Its eigenstates are simply plane waves:

$$\Psi_P(X) = e^{iP.X}, \tag{33}$$

with X the center of mass coordinates. The magnetic translations commutations (27) arise because the plane waves are extended bound state and not point particles as we have seen on this simple example.

The momentum $\vec{P}$ and the relative coordinate are then related by $-P_X = y_1 - y_2,\ P_Y = x_1 - x_2$ so that the bound state behaves as a neutral dipole with a dipole vector perpendicular and proportional to its momentum. Note that since the strength k of the spring enters the Hamiltonian (32) as a normalization factor the wave function (33) which describes the two charges is independent of k.

In fact, it is a general fact, if we replace the spring by a rotation invariant potential $V(r)$, one can refine the preceding approach to show that the wave function of the bound state is *independent* of the potential. To see it we need to consider the problem of the particle and the hole interacting in their respective LLL.

We mention that the recent developments in noncommutative string theory have the same origin. Indeed, one of the fundamental fields of string theory called the B field is the exact analogue of the magnetic field. When this field acquires an expectation value, the open strings behave very much like a spring in an external magnetic field [37].

3.2 Structure of the bound state

It is instructive to consider the dynamics of two particles within the lowest Landau level. The two particles interact through a potential $V(\mathbf{x_1} - \mathbf{x_2})$ which is supposed to be both translation and rotation invariant. In a physical situation the potential is the Coulomb interaction between the electrons, but it can in principle be any potential. For reasons that will become clear in the text we consider particles with a charge respectively equal to q_1 and q_2. Our aim is to show that, to a large extent, the properties of the dynamics are independent of the detailed shape of the potential. More precisely, the potential interaction is a two body operator which can be projected into the lowest Landau level. The projection consists in replacing the coordinates, $\mathbf{x_1},\ \mathbf{x_2}$, with the guiding center coordinates, $\mathbf{R_1},\ \mathbf{R_2}$. After the projection is taken, the potential becomes an operator which is the effective Hamiltonian for the lowest Landau level dynamics. Using a simple invariance argument, we can see that the eigenstates of the potential do not depend on it, as long as it is invariant under the isometries of the plane. In other words, the two body wave functions of the Hall effect are *independent* of the interactions. The case of the spring studied in the preceding section corresponds to $q_1 = -q_2$.

The guiding center coordinates for a particle of charge $q > 0$ times the charge of the electron have the expression:

$$b^+ = \frac{1}{\sqrt{2}}(-\partial_{\bar{z}} + qz), \quad b = \frac{1}{\sqrt{2}}(\partial_z + q\bar{z}). \tag{34}$$

Together with the angular momentum, $L = z\partial_z - \bar{z}\partial_{\bar{z}}$, they generate a central extension of the algebra of the isometries of the plane:

$$[b, b^+] = q, \quad [L, b^+] = b^+, \quad [L, b] = -b. \tag{35}$$

This algebra commutes with the Hamiltonian H, and therefore acts within the lowest Landau level. It plays a role similar to the angular momentum in quantum mechanics, and the operators b, b^+, L are the analogous of the angular momentum operators J^-, J^+, J^z . The Landau level index n plays the same role as the representation index j in the rotation group, and it can be recovered as the eigenvalue of a Casimir operator: $C = 2b^+b/q + L$. The states within each Landau level can be labeled by their angular momentum $m \leq n$.

When two particles of positive charge q_1 and q_2 are restricted to their respective lowest Landau level, we can form the operators $b^+ = b_1^+ + b_2^+$, $b = b_1 + b_2$ and the total angular momentum $L = L_1 + L_2$. These operators obey the commutation relations of the algebra (35) with the charge $q = q_1 + q_2$. Thus, as for the angular momentum, a product of two representation decomposes into representations of the isometry of the plane (35). The physically interesting case is when the two charges are equal to the electron charge ($q_1 = q_2 = 1$). It is easy to verify that each representation is constructed from a generating state state annihilated by b: $(b_1^+ - b_2^+)^n|0\rangle$, and the value of the Casimir operator is $C = -n$. The corresponding wave functions are:

$$\Psi_n(\bar{z}_1, \bar{z}_2) = (\bar{z}_1 - \bar{z}_2)^n \exp -(\bar{z}_1 z_1 + \bar{z}_2 z_2)/4l^2, \tag{36}$$

an expression that plays an important role in the theory of the fractional Hall effect. The potential being invariant under the displacements, it is a number V_n in each representation. Conversely, the $V_n' s$ are *all* the information about the potential that is retained by the lowest Landau level physics. The numbers V_n are called pseudopotentials, and turn out to be extremely useful to characterize the different phases of the fractional Hall effect [27].

When the two particles have charges of opposite sign, $q_1 > 0$ and $q_2 < 0$, $|q_2| < q_1$. Because of the sign of the second charge, b_2^+ and b_2 become respectively annihilation and creation operators and the lowest Landau level wave functions are polynomials in z_2 instead of $\bar{z}_2$. The same analysis can be repeated, but now the Casimir operator has a positive value n exactly as for the Landau levels. The physical interpretation is that a couple of charges with opposite sign behaves exactly like a bound state of charge $q^* = q_1 - |q_2|$. The states annihilated by b have a wave function independent of the precise expression of the potential, given by:

$$\Psi_n(\bar{z}_1, z_2) = z_2^n \exp\left(-q_1 \bar{z}_1 z_1/4l^2 - |q_2| \bar{z}_2 z_2/4l^2 + |q_2| \bar{z}_1 z_2/2l^2\right), \tag{37}$$

and they are the n^{th} Landau level's wave functions with the largest possible angular momentum $L = n$.

3.3 Application to $\nu = 1/2$

Let us see indicate how a scenario involving these composite particles enables to apprehend the Q.H.E. plateaux in the region of magnetic field around $\nu_0 = 1/2$. It could also apply to Bosonic particles interacting repulsively in a magnetic field at a filling factor $\nu = 1$. Although both problems first look different, they are essentially the same.

The LLL particles interact with a repulsive potential $V(\vec{x}-\vec{y})$. After projection the Hamiltonian takes the form:

$$H = 1/2 \int \rho(\vec{x})\ V(\vec{x}-\vec{y})\ \rho(\vec{y})\ d^2x\ d^2y \tag{38}$$

where $\rho(\vec{x})$ is the projected density operator. The projection relates a field $\rho(\vec{x})$ to a matrix $\hat{\rho}$ as we saw earlier, therefore this is a problem of *Matrix* quantum mechanics. If we decompose the field $\rho(x)$ into plane waves $\rho(x) = \sum_{\vec{k}} e^{ikx} \rho_k$, the plane waves obey the magnetic translation relations (27). Therefore, this Hamiltonian is not trivially diagonal as it would be in the absence of magnetic field.

The difficulty comes from the frustrated statistics. Had the particles the opposite statistics, it would be straightforward to find a good wave function for the ground state of (38). The approach proposed in [22], developed in [23][24][26][25] is to build the theory of these particles on top of the ground state having the opposite statistics. The "composite" particle is made of the original particle and a "hole' in the ground state. At $\nu = 1/2$ exactly, it is therefore a neutral fermionic bound state and the noncommutative theory can apply. Let us give a very sketchy description of such a theory.

The ground state is filled with bound-states introduced above made of an electron of charge -1, and a hole in a ground state filled with charges q. The charge of the bound state is thus $-q^*$, with:

$$q^* = 1 - q. \tag{39}$$

To obtain the values of the filling factor that give rise to a Hall effect, we use the fact (easy to verify) that for a fixed magnetic field and a fixed density, the following proportionality relation between the charge and the filling factor holds:

$$\text{charge} \propto \frac{1}{\text{filling factor}}, \tag{40}$$

When the magnetic field is varied, we take as a postulate that the charge q adjusts itself so that the filling factor of the ground state of the q charge is always equal to $1/2$. Thus, $q \propto 2$. An integer quantum Hall effect will develop when the filling factor of the bound state is an integer p. So, when $q* \propto 1/p$. We can recover the normalization coefficient through the relation between the filling factor of the electrons and their charge: $1 \propto 1/\nu$. Substituting these relations in (39) we obtain the following expression for the filling factors giving rise to a Hall effect:

$$\frac{1}{\nu} = 2 + \frac{1}{p}. \tag{41}$$

These filling factors are those predicted by Jain [10], and fit well with the Hall effect observed at $\nu = 3/7, 4/9, 5/11, 6/13$. In the region, close to $\nu = 1/2$, the quasiparticles have practically zero charge, and therefore see a weak magnetic field. They behave very much like a neutral Fermi liquid. This has been confirmed by several experiments [11]. One of them measures directly the charge q^* of the quasiparticles through the cyclotron radius of their trajectory [28]. At $\nu = 1/2$ exactly, the quasiparticles are neutral dipoles with a dipole size of the order of the magnetic length. The main difficulty for the theory is that the separation between two dipoles is of the same order as their size, and this is therefore a strong interaction problem. It would be extremely interesting if the new developments in NCFT [6] can help to make progress in this theory. It is not difficult to see that within this approach, and in the in the case of a δ potential interaction, the NCFT of the $\nu = 1/2$ state is the *noncommutative Gross-Neveu model.*

4 Skyrmion and non linear σ-model

As another application, we review the Skyrmion of the Hall effect [30] which relates the topological charge of the classical soliton to the electric charge of the quantum state. In a given topological sector, the solitons which minimize the action are in one to one correspondence with the degenerate eigenstates of a quantum Hamiltonian in the same charge sector. Moreover in both cases, the energy is equal to the modulus of the charge.

Belavin and Polyakov [31] have considered the classical solutions of the two dimensional nonlinear σ-model on the sphere S_2. The field configurations $\vec{n}(x, y)$ can be characterized by their stereographic projection on the complex plane $w(x, y)$. One requires that the spin points in the x direction at infinity, or equivalently $w(\infty) = 1$. The minima of the action are rational fractions of $z = x + iy$: $w(z) = f(z)/g(z)$. The soliton 's winding number and its classical action are both given by the degree k of the polynomials f, g. The soliton is thus determined by the positions where the spin points to the south and the north pole given by the zeros of f and g.

In quantum mechanics, the wave function for a single spin 1/2 particle constrained to the Lowest Landau Level (LLL) is fully determined by the positions where the spin is up or down with probability one. Up to an exponential term, the two components of this wave function are polynomials in z vanishing at the positions where the spin is respectively up and down. The wave functions:

$$\langle z_1, .., z_{N_e} | \Phi \rangle = \prod_{i=1}^{N_e} (f(z_i) \uparrow + g(z_i) \downarrow) \prod_{i<j} (z_i - z_j)^m \tag{42}$$

have been considered (for $m = 1$) by MacDonald, Fertig and Brey [33] in the Hall effect context. They represent the ground states of spin one half particles at a filling fraction close to $1/m$. Here we show that the correspondence with the nonlinear σ-model can be made precise in the case $m = 1$. One has the following correspondence table:

$\int (\vec{\nabla}\vec{n})^2 d^2x$	$\sum_{i<j} \delta^{(2)}(\vec{x}_i - \vec{x}_j)$
$\vec{n}(\vec{x})$	$\|\Phi\rangle$ a Slater determinant
winding number	electric charge

4.1 Non-commutatative Belavin-Polyakov soliton

Let us show that the Skyrmion is the exact non-commutative analogue of the Belavin-Polyakov soliton [32].

A point on the sphere S^2 is a unit vector $\vec{n}(\vec{x})$ with which we construct the projector $p(\vec{x}) = (1 + \vec{n}\vec{\sigma})/2$. p is a two by two rank one hermitian projector $p^2 = p,\ p^+ = p$ and the action for the non linear σ-model takes the form:

$$S = \frac{1}{\pi} \int \mathrm{tr}\ \partial_z p \partial_{\bar{z}} p\ d^2x \tag{43}$$

To obtain the solitons which minimize the action let us substitute $\partial_z p^2$ for $\partial_z p$ to rewrite the integrand as tr $p(\partial_z p \partial_{\bar{z}} p + \partial_{\bar{z}} p \partial_z p)$ and add to (43) the topological term:

$$K = \frac{1}{\pi} \int \text{tr } p(\partial_z p \partial_{\bar{z}} p - \partial_{\bar{z}} p \partial_z p) \; d^2x \tag{44}$$

so that the sum takes the form:

$$S' = S + K = \frac{2}{\pi} \int \text{tr } (p\partial_{\bar{z}} p)^+ (p\partial_{\bar{z}} p) \; d^2x \tag{45}$$

(45) is positive and the solutions with $S' = 0$ must obey $p\partial_{\bar{z}} p = 0$. If we parameterize p by a unitary vector v, $v^+v = 1$, $p = vv^+$, it is solved for $v = N^{-1}(f(z), g(z))$ where f, g are holomorphic functions and $N = \sqrt{|f|^2 + |g|^2}$. If one requires that $p(\infty) = (1 + \sigma_x)/2$, f and g are polynomials with the same highest coefficient z^k. The integrand of K is the field strength of the gauge potential $\omega = -v^+dv/2i$ which goes to a pure gauge far from soliton. The topological term is therefore given by the contour integral of ω at infinity equal to $-k$ and thus $S = k$.

The quantum analogous problem we consider here consists in finding the degenerate ground states of electrons interacting by a δ repulsive potential in the lowest Landau level (LLL). The electrons are confined in a finite disc thread by N_ϕ magnetic fluxes. When the number of electrons N_e differs from N_ϕ by an integer equal to the winding number k the quantum eigenstates coincide with the classical solitonic field configurations if the scale of variation of the soliton is large compared to the magnetic length.

The second quantized field that annihilates (creates) an electron with a spin σ at position $\vec{x}$ in the LLL can be constructed in terms of the fermionic operators $c_{l\sigma}$ ($c^+_{l\sigma}$) which annihilate (create) an electron in the l^{th} orbital:

$$\Psi_\sigma(\vec{x}) = \sum_l \langle z|l\rangle c_{l\sigma} \tag{46}$$

In terms of this field, the total number of electrons in the LLL is $N_e = \int \sum_\sigma \Psi^+_\sigma \Psi_\sigma(\vec{x})$ d^2x. The charge of the Skyrmion is the difference between the number of magnetic fluxes N_ϕ and the number of electrons N_e: $Q_s = N_\phi - N_e$. In other words, it is the number of electrons added or subtracted to the system starting from a situation where the filling factor $\nu = N_e/N_\phi$ is exactly one. In the following we consider the limit $N_\phi,\ N_e = \infty$ keeping the charge Q_s fixed.

In [33], it was observed that the zero energy states of the hard-core model Hamiltonian could be completely determined. We consider a closely related short range repulsive Hamiltonian invariant under a particle hole transformation $\Psi \to \Psi^+$ and such that the energy of its ground state coincides with the charge. It is given by:

$$H = \frac{1}{\rho} \int (\Psi^+_\uparrow \Psi_\uparrow - \Psi_\downarrow \Psi^+_\downarrow)^2(\vec{x}) \; d^2x = \frac{2}{\rho} \int (\Psi_\uparrow \Psi_\downarrow)^+ (\Psi_\uparrow \Psi_\downarrow)(\vec{x}) \; d^2x + Q_s \tag{47}$$

where we have used the fact that $\{\Psi_\sigma(\vec{x}), \Psi^+_{\sigma'}(\vec{x})\} = \rho\delta_{\sigma\sigma'}$ to obtain the second equality. Let us for simplicity consider the case where $Q_s > 0$, the other case can be reached using a particle hole transformation. This Hamiltonian is clearly bounded from bellow by Q_s and the exact eigenstates with energy Q_s are obtained for states $|\Phi\rangle$ such that $\Psi_\uparrow(\vec{x})\Psi_\downarrow(\vec{x})|\Phi\rangle = 0$. In such a state two electrons never occupy the same position and the wave function is blind to the short range potential. This property is precisely guaranteed by the factor $\prod_{i<j}(z_i - z_j)$ in (42).

The states (42) carry a charge $Q_s = k$ where k is the degree of the polynomials f and g. They are Slater determinants:

$$\langle z_1, .., z_{N_e}|\Phi\rangle = \bigwedge_1^{N_e}(f(z_i)\uparrow + g(z_i)\downarrow)\langle z_i|\tilde{l}\rangle \tag{48}$$

where $\langle z|\tilde{l}\rangle$ is basis of orthogonal polynomials for the scalar product:

$$\langle\phi|\phi'\rangle = \int \bar{\phi}(\bar{z})\phi'(z)e^{-q\bar{z}z}(|f|^2 + |g|^2)d^2x \tag{49}$$

A Slater determinant is fully determined by the matrix expectation value:

$$\langle z|\hat{P}|z\rangle = \rho p = \langle\Phi| \begin{pmatrix} \Psi_\downarrow^+\Psi_\downarrow & \Psi_\uparrow^+\Psi_\downarrow \\ \Psi_\downarrow^+\Psi_\uparrow & \Psi_\uparrow^+\Psi_\uparrow \end{pmatrix} |\Phi\rangle \tag{50}$$

and $\hat{P}$ is a projector $\hat{P}^2 = \hat{P}$. In the case of (48) we can obtain $\hat{P}$ explicitly as follows. The states $|v_l\rangle = (f(b^+)\uparrow + g(b^+)\downarrow)|\tilde{l}\rangle$ can be organized into a vector $V = \sum_l |v_l\rangle\langle l|$. By construction, $V^+V = \mathrm{Id}$, so that, $VV^+ = \hat{P}$, where $\hat{P}$ is the projector with Q-symbol p.

To relate the Skyrmion to the classical σ-model, let us evaluate the energy of a slater determinant $|\Phi\rangle$ using the Wick theorem:

$$\langle\Phi|H|\Phi\rangle = \rho\int(2\det p - \mathrm{tr}p + 1)\ d^2x \tag{51}$$

In the above expression, the determinant is evaluated using the ordinary product. Suppose we replace it with the $*$-product in (51). Using the fact tat $p * p = p$ one verifies that the integrand rewrites $(\mathrm{tr}p - 1) * (\mathrm{tr}p - 1)$. Since $\mathrm{tr}(p-1)$ is $\mathrm{O}(\rho^{-1})$, the *-square is $\mathrm{O}(\rho^{-2})$ and does not contribute to the energy when $\rho \to \infty$. Therefore, the limiting value of (51) is given by the modification induced by the ordinary product at first order in ρ^{-1}. One obtains from (28):

$$\langle\Phi|H|\Phi\rangle = \frac{1}{\pi}\int \mathrm{tr}\ \partial_z p\partial_{\bar{z}}p\ d^2x + O(1/\rho) \tag{52}$$

which is the value of the action (43) and establishes the correspondence between the classical and the quantum problems.

Although the classical action (43) can be obtained straightforwardly from the energy (51) in the limit $\rho \to \infty$, the topological term (44) cannot so directly be related to the charge of the Skyrmion. It is nevertheless possible to define the topological term at the quantum level and to verify it coincides with the charge in the present case. For this we need to make the substitutions (29) in (44):

$$p \to \hat{P}, \quad \frac{1}{\pi}\int\ .\ d^2x \to \frac{1}{q}\mathrm{Tr}\ ., \quad \partial_{\mathrm{z}}. \to \mathrm{q}[\mathrm{b}, .], \quad \partial_{\bar{\mathrm{z}}}. \to -\mathrm{q}[\mathrm{b}^+, .] \tag{53}$$

The modified expression of K (44) still defines a topological invariant which is a noncommutative analogue of K [36] to which it reduces in the limit $\rho \to \infty$. It can be defined for projectors $\hat{P}$ which do not have a classical limit p. The easiest way to realize a charge $-k$ configuration consists in expelling k electrons from the first $l < k$ angular momentum orbitals in the $\nu = 1$ filled LLL (Here the spin can be kept fixed and plays no essential role). The projector which characterizes this configuration is $\hat{P} = \sum_{l\geq k}|l\rangle\langle l|$, equivalently $V = b^{+k}N^{-1} = \sum_l |l+k\rangle\langle l|$. Using the quantum expression one can easily verify that the topological invariant is equal to the charge $K = -k$.

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